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Study on Fixed Point Theorems in Complete Cone Metric Spaces

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ABSTRACT: In this paper we established some fixed point theorems of contractions functions in complete cone metric spaces.

KEYWORDS: Metric space, cone metric space, fixed point, complete cone metric space, Cauchy sequence.

I. INTRODUCTION

The Cone Metric space is ageneralization of metric spaces. The notion of cone metric space is initiated by Huang and Zhang [1] and also they discussed some properties of the convergence of sequences and proved the fixed point theorems of a contraction mapping in cone metric spaces. Many authors have studied the existence and uniqueness of fixed points for single valued mappings and multivalued mappings in metric spaces. In this paper we discuss existence and unique fixed point in complete normal cone metric spaces, which are the generalization of some existing theorems.

DEFINITION 1.1: A subset *P* of *E* is called a cone if and only if

- 1. *P* is closed, nonempty and $P \neq 0$
- 2. $ax + by \in P$ for all $x, y \in P$ and nonnegative real numbers a, b
- 3. $P \cap P^- = \{0\}.$

Given a cone $P \subset E$, we define a partial ordering \leq with respect to P by $x \leq y$ if and only if $y - x \in P$. We will write x < y to indicate that $x \leq y$ but $x \neq y$, while x, y will stand for $y - x \in intP$, where intP denotes the interior of P. The cone P is called normal if there is a number K > 0 such that $0 \leq x \leq y$ implies $||x|| \leq K ||y||$ for all $x, y \in E$. The least positive number satisfying the above is called the normal constant. The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}$ is sequence such that $x_1 \leq x_2 \leq \cdots \leq x_n \ldots \leq y$ for some $y \in E$, then there is $x \in E$ such that $||x_n - x|| \to 0$ as $n \to 0$. Equivalently the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone. Suppose E is a Banach space, P is a cone in E with $intP \neq 0$ and \leq is partial ordering with respect to P.

EXAMPLE:Let K > 1 be given. Consider the real vector space with

$$E = \left\{ ax + b; a, b \in \mathbb{R}; x \in \left[1 - \frac{1}{k}, 1\right] \right\}$$

With supremum norm and the cone $P = \{ax + b : a \ge 0, b \le 0\}$ in *E*. The cone *P* is regular and so normal.



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DEFINITION 1.2: Suppose that *E* is a real Banach space, then *P* is a cone in *E* with $intp \neq \emptyset$, and \leq is partial ordering with respect to *P*. Let *X* be a nonempty set, a function $d: X \times X \to E$ is called a cone metric on *X* if it satisfies the following conditions with

- 1. $d(x, y) \ge 0$, and d(x, y) = 0 if and only if $x = y \forall x, y \in X$
- 2. $d(x, y) = d(y, x), \forall x, y \in X$
- 3. $d(x, y) \le d(x, z) + d(z, y), \forall x, y, z \in X$

Then (X, d) is called a cone metric space (CMS)

EXAMPLE:Let = \mathbb{R}^2 , $P = \{(x, y) : x, y \ge 0\}$ $X = \mathbb{R}$ and $d: X \times X \to E$ such that

$$d(x, y) = (|x - y|, \alpha | x, y|)$$

Where $\alpha \ge 0$ is a constant. Then (X, d) is a cone metric space.

DEFINITION 1.3: Let (X, d) be a CMS and $\{x_n\}_{n\geq 0}$ be a sequence in X. Then $\{x_n\}_{n\geq 0}$ converges to x in X whenever for every $c \in E$ with $0 \ll c$, there is a natural number $N \in N$ such that $d(x_n, x) \ll c$ for all $n \ge N$. It is denoted by $\lim_{n\to\infty} x_n = x \operatorname{orx}_n \to x$.

DEFINITION 1.4: Let (X, d) be a CMS and $\{x_n\}_{n\geq 0}$ be a sequence in X. Then $\{x_n\}_{n\geq 0}$ is a cauchy sequence whenever for every $c \in E$ with $0 \ll c$, there is a natural number $N \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.

LEMMA 1.5: Let (X, d) be a cone metric space, P is a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x and $\{x_n\}$ converges to y, then x = y.

That is the limit of $\{x_n\}$ is unique.

DEFINITION 1.6: Let (X, d) be a cone metric space, if every Cauchy sequence is convergent in X, then X is called a complete cone metric space.

LEMMA 1.7: Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$, $n, m \to \infty$.

II. MAIN RESULT

THEOREM 2.1: Let (X, d) be a complete cone metric space and P be a normal cone with constant K. Suppose the mapping $T: X \to X$ satisfies the following conditions:

$$d(Tx,Ty) \le \left(\frac{d(y,Ty)d(x,Tx)}{J}\right) + I$$

For all $x, y \in X$, where IJ = d(x, Tx)d(x, y) & J = d(x, Tx) + d(y, Ty) + l, $J \ge 1, l \ge 1$. Then

- (i) T has unique fixed point in X.
- (ii) $T^n x'$ Converges to a fixed point, for all $x' \in X$.

Proof: (i) Let $x_0 \in X$ be arbitrary and choose a sequence $\{x_n\}$ such that $x_{n+1} = Tx_n$. $d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1})$



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$$\leq \left(\frac{d(x_{n-1}, Tx_{n-1}) + d(x_n, x_{n-1})}{J_n}\right) + I_n$$

$$\leq \left(\frac{d(x_{n-1}, x_n)}{J_n} + \frac{I_n}{d(x_{n-1}, x_n)}\right) d(x_n, x_{n-1})$$

Take

$$\lambda_n = \left(\frac{d(x_{n-1}, x_n)}{J_n} + \frac{I_n}{d(x_{n-1}, x_n)}\right)$$

We have

$$d(x_{n+1}, x_n) \leq \lambda_n d(x_n, x_{n-1})$$

$$\leq (\lambda_n \lambda_{n-1}) d(x_{n-1}, x_{n-2})$$

$$\vdots$$

$$\leq (\lambda_n \lambda_{n-1} \dots \lambda_1) d(x_1, x_0).$$

Observe that (λ_n) is non increasing, with positive terms. So, $\lambda_1 \dots \lambda_n \leq \lambda_1^n \text{ and } \lambda_1^n \to 0$. It follows that $\lim_{n\to\infty} \lambda_1 \lambda_2 \dots \lambda_n = 0$.

Thus, it is verified that, $\lim_{n \to \infty} d(x_{n+1}, x_n) = 0.$

Now for all $m, n \in \mathbb{N}$ and m > n we have

$$\begin{aligned} d(x_{m}, x_{n}) &\leq d(x_{n}, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots + d(x_{m-1}, x_{m}) \\ &\leq [(\lambda_{n}\lambda_{n-1}\dots\lambda_{1}) + (\lambda_{n+1}\lambda_{n}\dots\lambda_{1}) + \dots + (\lambda_{m-1}\lambda_{m-2}\dots\lambda_{1})]d(x_{1}, x_{0}) \\ &= \sum_{k=n}^{m-1} (\lambda_{k}\lambda_{k-1}\dots\lambda_{1})d(x_{1}, x_{0}) \\ \|d(x_{m}, x_{n})\| &\leq K \|\sum_{\substack{k=n \\ m-1}}^{m-1} (\lambda_{k}\lambda_{k-1}\dots\lambda_{1})d(x_{1}, x_{0})\| \\ \|d(x_{m}, x_{n})\| &\leq K \sum_{\substack{k=n \\ m-1}}^{m-1} (\lambda_{k}\lambda_{k-1}\dots\lambda_{1})\| d(x_{1}, x_{0})\| \\ \|d(x_{m}, x_{n})\| &\leq K \sum_{\substack{k=n \\ m-1}}^{m-1} a_{k} \|d(x_{1}, x_{0})\|, \end{aligned}$$

Where $a_k = (\lambda_k \lambda_{k-1} \dots \lambda_1)$ and K is normal constant of P. Now $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} < 1$ and $\sum_{k=1}^{\infty} a_k$ is finite, and $\sum_{k=n}^{m-1} (\lambda_k \lambda_{k-1} \dots \lambda_1) \to 0$, as $m, n \to \infty$.

Hence $\{a_k\}$ is convergent by D' Alembert's ratio test, Therefore $\{x_n\}$ is a Cauchy sequence. There is $x' \in X$ such that $x_n \to x'$ as $n \to \infty$.

$$d(Tx', x') \leq d(Tx', Tx_n) + d(Tx_n, x')$$

$$\leq \left(\frac{d(x', Tx')d(x_n, x')}{J_n} + I_n\right) + d(Tx_n, x')$$

$$\leq \left(\frac{d(x', Tx')d(x_n, x')}{J_n} + \frac{I_n}{d(x_n, x')}\right) d(x_n, x') + d(x_{n+1}, x')$$

 $d(Tx', x') \leq 0 asn \to \infty$

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Therefore || d(x', Tx') || = 0. Thus, x' is a fixed point of T.

UNIOUENESS

Suppose x' and y' are two fixed points of T.

d(x', y') = d(Tx', Ty') $\leq \left(\frac{\dot{d}(x', Tx')}{J} + \frac{I}{d(x', y')}\right) d(x', y') \leq 0$

Therefore ||d(x', y')|| = 0. Thus x' = y'. Hence x' is an unique fixed point of T.

(ii)Now

 $d(T^{n}x',x') = d(T^{n-1}(Tx'),x') = d(T^{n-1}x',x') = d(T^{n-2}(Tx'),x') \dots = d(Tx',x') = 0$ Hence $T^n x'$ converges to a fixed point, for all $x' \in X$.

COROLLARY 2.2: Let (X, d) be a complete cone metric space and P be a normal cone with normal constant K. suppose the mapping $T: X \to X$ satisfies the following conditions:

$$d(Tx,Ty) \leq \left(\frac{d(y,Ty)}{J} + \frac{I}{d(x,y)}\right)d(x,y)$$

For all $x, y \in X$. Where IJ = d(x, Tx)d(x, y) & J = d(x, Tx) + d(y, Ty) + 1. Then

- (i) T has unique fixed point in X.
- (ii) $T^n x'$ Converges to a fixed point, for all $x' \in X$.

Proof: The proof of the corollary immediate by taking l = 1 in the above theorem.

THEOREM 2.3: Let (X, d) be a complete metric space and let T be a mapping from X into itself. Suppose that T satisfies the following condition:

$$d(Tx,Ty) \leq \left(\frac{d(y,Ty)d(x,y)}{J}\right)$$

For all $x, y \in X$, where $J = d(x, Tx) + d(y, Ty) + l, l \ge 1$. Then

- (i) T has unique fixed point in X.
- (ii) $T^n x'$ Converges to a fixed point, for all $x' \in X$.

Proof: (i) Let $x' \in X$ be arbitrary and choose a sequence $\{x_n\}$ such that $x_{n+1} = Tx_n$. We have

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \\ \leq \left(\frac{d(x_{n-1}, Tx_{n-1})}{J_n}\right) d(x_n, x_{n-1}) \\ \leq \left(\frac{d(x_{n-1}, x_n)}{J_n}\right) d(x_n, x_{n-1})$$

Take

Take

$$\lambda_{n} = \left(\frac{d(x_{n-1}, x_{n})}{J_{n}}\right),$$

We have
$$d(x_{n}, x_{n+1}) \leq \lambda_{n} d(x_{n-1}, x_{n})$$
$$\leq (\lambda_{n} \lambda_{n-1}) d(x_{n-1}, x_{n-2})$$
$$\vdots$$
$$\leq (\lambda_{n} \lambda_{n-1} \dots \lambda_{1}) d(x_{1}, x_{0}).$$



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Observe that $\{\lambda_n\}$ is non increasing, with positive terms. So $\lambda_1 \dots \lambda_n \leq \lambda_1^n \operatorname{and} \lambda_1^n \to 0$. It follows that $\lim_{n \to \infty} (\lambda_1 \lambda_2 \dots \lambda_n) = 0$. Thus, it is verified that $\lim_{n \to \infty} d(x_{n+1}, x_n) = 0$ Now for all $m, n \in \mathbb{N}$ we have $d(x_m, x_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots + d(x_{m-1}, x_m)$ $\leq [(\lambda_n \lambda_{n-1} \dots \lambda_1) + (\lambda_{n+1} \lambda_n \dots \lambda_1) + \dots + (\lambda_{m-1} \lambda_{m-2} \dots \lambda_1)]d(x_1, x_0)$ $= \sum_{k=n}^{m-1} (\lambda_k \lambda_{k-1} \dots \lambda_1) d(x_1, x_0)$ $\|d(x_m, x_n)\| \leq K \sum_{k=n}^{m-1} a_k \|d(x_1, x_0)\|$ $\|d(x_m, x_n)\| \leq K \sum_{k=n}^{m-1} a_k \|d(x_1, x_0)\|$

Where $a_k = \lambda_k \lambda_{k-1} \dots \lambda_1$ and K is normal constant of P. Now $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} < 1$ and $\sum_{k=1}^{\infty} a_k$ is finite, and $\sum_{k=n}^{m-1} (\lambda_k \lambda_{k-1} \dots \lambda_1) \to 0$, as $m, n \to \infty$.

Hence $\{a_k\}$ is convergent by D' Alembert's ratio test, Therefore $\{x_n\}$ is a Cauchy sequence. There is $x' \in X$ such that $x_n \to x'$.

$$d(Tx', x') \leq d(Tx', Tx_n) + d(Tx_n, x')$$

$$\leq \left(\frac{d(x_n, Tx_n)}{J_n}\right) d(x_n, x') + d(Tx_n, x')$$

$$\leq \left(\frac{d(x_n, x_{n+1})}{J_n}\right) d(x_n, x') + d(x_{n+1}, x')$$

$$d(Tx', x') \leq 0 \text{as} n \to \infty$$

Therefore ||d(x', Tx')|| = 0. Thus Tx' = x'.

UNIQUENESS

Suppose x' and y' are two fixed points of T. d(x', y') = d(Tx', Ty')

$$\leq \left(\frac{d(y', \top y')d(x', y')}{J}\right) \leq 0$$

Therefore|| d(x', y')|| = 0. Thus x' = y'. Hence x' is an unique fixed point of T. (ii) Now $d(T^n x', x') = d(T^{n-1}(Tx'), x') = d(T^{n-1}x', x') = d(T^{n-2}(Tx'), x') \dots = d(Tx', x') = 0$ Hence $T^n x'$ converges to a fixed point, for all $x' \in X$.

COROLLARY 2.4: Let (X, d) be a complete metric space and let T be a mapping from X into itself. Suppose that T satisfies the following condition:

 $d(Tx, Ty) \leq \left(\frac{d(y, Ty)d(x, y)}{J}\right)$ For all $x, y \in X$, where $J = d(x, Tx) + d(y, Ty) + l, l \geq 1$. Then (i) T has unique fixed point in X. (ii) $T^n x'$ Converges to a fixed point, for all $x' \in X$.

Proof: The proof of the corollary immediate by taking l = 1 in the above theorem.



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III. CONCLUSION

We have proved some fixed point theorems of contraction functions in complete cone metric spaces. The presented results generalization of some existing results.

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